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Slab Percolation and Phase Transitions for the Ising Model

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Abstract

We prove, using the random-cluster model, a strict inequality between site percolation and magnetization in the region of phase transition for the d-dimensional Ising model, thus improving a result of [CNPR76]. We extend this result also at the case of two plane lattices $\mathbb{Z}^2$ (slabs) and give a characterization of phase transition in this case. The general case of N slabs, with N an arbitrary positive integer, is partially solved and it is used to show that this characterization holds in the case of three slabs with periodic boundary conditions. However in this case we do not obtain useful inequalities between magnetization and percolation probability.

Keywords: Percolation, infinite clusters, magnetization, Gibbs measure, random-cluster measure.

1 Introduction

At the end of the 70's the seminal paper [CNPR76] showed the connection between phase transition for the ferromagnetic Ising model and site percolation. This point of view has given a geometrical interpretation of phase transition, initiating a new line of research. Following this approach Higuchi developed techniques to study percolation for the two dimensional Ising model, with non zero external field [Hig85, Hig87, Hig93a, Hig93b]. For example, he showed that for every $\beta < \beta_c$ there exists a positive critical point $h_c(\beta)$ such that an infinite cluster of $(+)$-sites does not exist for all $h < h_c$. Russo [Rus79], in 1979, proved that if an equilibrium Ising measure is invariant under translations along one direction of the two dimensional lattice then it is invariant under all translations. Hence, Aizenman and Higuchi (see [Aiz80, Hig81] and also the more recent paper [GH00]) showed that the only extremal Gibbs measures are μ_+ and μ_-. In three dimensions the situation is different: Dobrushin showed that at low temperatures there exist non-translation invariant Gibbs states. There are substantial differences between two and

three dimensions also for percolation for the Ising measure; in fact in [CR85] it is showed that in three dimensions there is coexistence of infinite plus and minus clusters (at least for small values of the parameter β), while in two dimensions it is proved that the infinite clusters of opposite sign can not coexist [CNPR76, GKR88].

The paper is organized as follows: in Section 2 we set the notation and introduce some basic tools. In Section 3 we use the random cluster model to give, below the critical temperature, a strict inequality between magnetization and site percolation probability for the d-dimensional cubic lattice, in this way we improve a result of [CNPR76]. Then, in Section 4 we partially generalize the result to some *slab graphs*. A slab graph is a graph $G_N = (V_N, E_N)$, where $V_N = \mathbb{Z}^2 \times \{0, \ldots, N-1\}$ and E_N is the set of all pairs of vertices in V_N having Euclidean distance equal to one. Such graphs, also called *bunkbed graphs*, have attracted the attention of other researches (see [BB97, Häg03]) in the study of random walks, random-cluster model and some correlation inequalities for the Ising model. In Section 5, for $N = 2$, we prove a characterization of phase transitions similar to that for the $\mathbb{Z}^2$ lattice, thereby obtaining an inequality between magnetization and percolation probability of *columns* formed only of $+1$ spins. For $N = 3$ and periodic boundary conditions, we are able to characterize the phase transition through percolation of columns with majority of plus. However we cannot obtain any meaningful inequality in this case.

Recently, Bodineau has proved a fine and natural result on slab percolation for the Ising model (see [Bod03]); let $\beta_c^{(N)}$ be the critical point for the graph G_N and $\beta_c(3)$ the critical point for the three dimensional cubic lattice. It is easy to show, using the FKG inequality for the random cluster measure, that $\beta_c^{(N)} > \beta_c^{(N+1)} > \cdots \geq \beta_c(3)$ for every N. However Bodineau also show that $\lim_{N\to\infty} \beta_c^{(N)} = \beta_c(3)$; thus the N slabs are a good approximation for the three dimensional Ising model, at least for the purpose of estimating the critical point.

2 Basic definitions and notation

In this section we set our notation for percolation, ferromagnetic Ising model, and random-cluster model.

Let $d \geq 2$ and let $\mathbb{Z}^d$ be the set of all d-vectors $x = (x_1, x_2, \ldots, x_d)$ with integral coordinates. The distance $||x - y||$ from x to y is defined by $||x - y|| = \sum_{i=1}^{d} |x_i - y_i|$. If $||x - y|| = 1$ we say that x and y are *adjacent*. We turn $\mathbb{Z}^d$ into a graph, called the *d-dimensional cubic lattice*, by adding edges $e = \langle x, y\rangle$ between all pairs x, y of adjacent points of $\mathbb{Z}^d$; we denote this lattice by $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d$ is the edge set. The edge $e = \langle x, y\rangle$ is said to be *incident* to the vertices x and y; in this case we also say that x and y are endvertices of the edge $e \in \mathbb{E}^d$.

A *path* of $\mathbb{L}^d$ is an alternating sequence $x_0, e_0, \ldots, e_{n-1}, x_n$ of distinct vertices and edges with $e_i = \langle x_i, x_{i+1}\rangle$ for all $i = 0, \ldots, n-1$; such a path has *length* n and is said to *connect* x_0 to x_n. A subset $Y \subset \mathbb{Z}^d$ is *connected* if for all pairs x, y of vertices in Y, there exists a path connecting the vertices x, y having all its vertices belonging to Y.

The *boundary* of $Y \subset \mathbb{Z}^d$ is the set ∂Y of all vertices in $\mathbb{Z}^d \setminus Y$ that are adjacent to at least one vertex in Y.

The *edge configuration space* is $\Omega = \{0,1\}^{\mathbb{E}^d}$, so its elements are vectors $\omega = (\omega(e) : e \in \mathbb{E}^d)$. We say that the edge e is *open* if $\omega(e) = 1$, and *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, we consider the random subgraph of $\mathbb{L}^d$ containing the vertex set $\mathbb{Z}^d$ and the open edges $(\omega^{-1}(1))$ only; an *open cluster* of ω is a maximal connected component of this graph.

Let $\Sigma = \{-1,+1\}^{\mathbb{Z}^d}$ be the *spin configuration space*, elements of which are $\sigma = (\sigma_x : x \in \mathbb{Z}^d)$. We say that the vertex x has spin $+1$ (-1) if $\sigma_x = +1$ (-1) . For $\sigma \in \Sigma$, we consider the random subgraph of $\mathbb{L}^d$ containing the edge set $\mathbb{E}^d$ and the vertices $\sigma^{-1}(+1)$ only; a $(+)$-*cluster* of σ is a maximal connected component of this graph. A $(-)$-*cluster* is defined in a similar way. We use the notation $(\infty, \pm)$-cluster to indicate an infinite $(\pm)$-cluster.

The spaces Ω and Σ are endowed with the discrete topology. We denote by $\mathcal{F}$ the σ-field generated by the finite-dimensional cylinders. For $A \in \mathcal{F}$, we indicate with $\bar{A}$ the complement of A.

Let Λ be a finite box of $\mathbb{Z}^d$, *i.e.* $\Lambda = \prod_{i=1}^{d}[x_i, y_i]$ for some $x, y \in \mathbb{Z}^d$, where $[x_i, y_i]$ is the set $\{x_i, x_i+1, x_i+2, \ldots, y_i\}$. We write $\mathbb{E}_\Lambda$ for the set of edges $e = \langle x, y\rangle$ in $\mathbb{E}^d$ such that $x, y \in \Lambda$ and we define

$$\Omega^1_\Lambda = \{\omega \in \Omega : \omega(e) = 1 \text{ for all } e \in \mathbb{E}^d \setminus \mathbb{E}_\Lambda\}, \tag{1}$$

$$\Sigma^+_\Lambda = \{\sigma \in \Sigma : \sigma_x = +1 \text{ for all } x \in \mathbb{Z}^d \setminus \Lambda\}. \tag{2}$$

Σ^-_Λ is defined analogously.

For $0 \le p \le 1$, let $\phi^1_{\Lambda,p}$ be the random-cluster measure on Ω^1_Λ with wired boundary conditions [Gri04], and let $\mu^\pm_{\Lambda,\beta,J}$ (or simply $\mu^\pm_\Lambda$) be the Ising Gibbs measure on $\Sigma^\pm_\Lambda$ with $(\pm)$-boundary conditions, zero external field ($h = 0$) and interactions $\{J_e\}_{e\in\mathbb{E}_{\Lambda\cup\partial\Lambda}}$[Lig85]. In this paper we agree that on each edge e of the graph under consideration there is a constant interaction $J_e \equiv J = 1$. In some cases one could take different values of the interactions on different edges; this will be partially discussed in the last section.

For $p = 1 - \exp\{-2\beta\}$, we define the *coupling between Ising and random-cluster measures* $\nu^+_{\Lambda,p}$ (or simply ν^+_Λ) on $\Sigma^+_\Lambda \times \Omega^1_\Lambda$ with $(+)$-boundary conditions, as

$$\nu^+_\Lambda(\sigma, \omega) = K \prod_{e\in\mathbb{E}_{\Lambda\cup\partial\Lambda}} \{(1-p)\delta_{\omega(e),0} + p\,\delta_{\omega(e),1}\delta_e(\sigma)\}, \tag{3}$$

where K is the normalizing constant, and $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ for $e = \langle x, y\rangle$ ($\delta_{i,j}$ is the Kronecker delta). Similarly, we define the coupling measure ν^-_Λ with $(-)$-boundary conditions. The marginal measure of ν^+_Λ on Σ^+_Λ is the Ising measure μ^+_Λ, and the marginal measure on Ω^1_Λ is the random-cluster measure $\phi^1_{\Lambda,p}$. We say that an open cluster of $\omega \in \Omega^1_\Lambda$ (or a $(+)$-cluster of $\sigma \in \Sigma^+_\Lambda$) *touches* $\partial\Lambda$ if at least one vertex of this cluster belongs to $\partial\Lambda$. Given ω, the conditional measure of ν^+_Λ on Σ^+_Λ is obtained by putting

$\sigma_x = +1$ for every $x \in \partial\Lambda$, then by setting $\sigma_x = +1$ for every x in an open cluster touching $\partial\Lambda$, and finally by choosing spins $+1$ or -1 with probability $\frac{1}{2}$ on all open clusters not touching $\partial\Lambda$. Given σ, the conditional measure of ν_Λ^+ on Ω_Λ^1 is obtained by setting $\omega(e) = 1$ for every $e \in \mathbb{E}_{\partial\Lambda}$, then for $e \in \mathbb{E}_\Lambda$, $\omega(e) = 0$ if $\delta_e(\sigma) = 0$ and $\omega(e) = 1$ with probability p (independently of other edges) if $\delta_e(\sigma) = 1$.

Using standard arguments on the stochastic order (FKG order) one can define these measures directly on the infinite spaces by taking the weak limit of measures; thus are well defined: $(\Sigma \times \Omega, \mathcal{F}, \nu_+)$, $(\Omega, \mathcal{F}, \phi_p^1)$ and $(\Sigma, \mathcal{F}, \mu_+)$, where ϕ_p^1, μ_+ and ν_+ are respectively the random-cluster measure on Ω with wired boundary conditions, the Ising measure on Σ with (+)-boundary conditions, and the coupling measure on $\Sigma \times \Omega$ with (+)-boundary conditions. Moreover for any given $\omega \in \Omega$, the conditional probability measure $\nu_+(\cdot|\omega)$ is obtained by setting $\sigma_x = +1$ for all the vertices x belonging to an infinite open cluster of ω, and by putting spins $+1$ or -1 with probability $\frac{1}{2}$ on each other cluster.

For more details and in the more general setting of every boundary condition and no-ferromagnetic interactions see [New97] in which all constructions and relations between the three measures ν_+, μ_+ and ϕ_p^1 are clearly explained.

We define $\tau_\Lambda = \mu_\Lambda^+(\sigma_0 = +1) - \frac{1}{2}$, where 0 denotes the origin of $\mathbb{Z}^d$. If $\tau = \lim_{\Lambda \uparrow \mathbb{Z}^d} \tau_\Lambda$ (the limit exists by monotonicity) then

$$\tau = \mu_+(\sigma_0 = +1) - \frac{1}{2}. \tag{4}$$

Given X, Y subsets of $\mathbb{Z}^d$ we denote with $\{X \leftrightarrow Y\}$ the set of configurations $\omega \in \Omega$ such that there exists a vertex $x \in X$ connected to a vertex $y \in Y$ by a path of open edges. We write $\{x \leftrightarrow \infty\}$ for the set of configurations $\omega \in \Omega$ such that x belongs to an infinite open cluster. The following Proposition is a result due to Kasteleyn and Fortuin (see [Gri04]).

Proposition 2.1. *If $p = 1 - \exp\{-2\beta\}$, then $\tau_\Lambda = \frac{1}{2}\, \phi_{\Lambda,p}^1(0 \leftrightarrow \partial\Lambda)$. Equality holds also in the limit $\Lambda \uparrow \mathbb{Z}^d$: $\tau = \dfrac{1}{2}\, \phi_p^1(0 \leftrightarrow \infty)$.*

We give some other definitions. We put $C_\infty^\pm = \{\sigma \in \Sigma : 0 \in (\infty, \pm)\text{-cluster of } \sigma\}$. The *percolation probability* is denoted by $R(\pm; \mu_\pm) = \mu_\pm(C_\infty^\pm)$, and the *magnetization in the origin* is

$$M(\mu_\pm) = \mathbb{E}_{\mu_\pm}(\sigma_0) = \mu_\pm(\sigma_0 = +1) - \mu_\pm(\sigma_0 = -1), \tag{5}$$

where σ_0 is the spin on the origin. By (4), (5) and Proposition 2.1 follows

$$M(\mu_+) = 2\tau = \phi_p^1(0 \leftrightarrow \infty). \tag{6}$$

3 $\mathbb{Z}^d$ percolation and magnetization

In this section we use the random-cluster model to prove, in a different way, the inequality relating percolation probability and magnetization in the d-dimensional Ising

model, given in [CNPR76]. Moreover, we prove that for $T < T_c(d)$ this inequality is strict. For $d = 2$ we have a complete characterization of phase transition in the Ising model through percolation.

Theorem 3.1. *For a ferromagnetic Ising model on* $(\Sigma, \mathcal{F}, \mu_\pm)$ *with zero external field, the following inequality holds:*

$$|M(\mu_\pm)| \leq R(\pm;\, \mu_\pm). \tag{7}$$

Proof. We consider the following event:

$$\Sigma \times \{\omega \in \Omega : 0 \leftrightarrow \infty\} = C_\infty^+ \times \{\omega \in \Omega : 0 \leftrightarrow \infty\} \cup \overline{C_\infty^+} \times \{\omega \in \Omega : 0 \leftrightarrow \infty\} \tag{8}$$

and we observe that

$$\nu_+(\overline{C_\infty^+} \times \{0 \leftrightarrow \infty\}) = 0. \tag{9}$$

Therefore

$$\phi_p^1(0 \leftrightarrow \infty) = \nu_+(\Sigma \times \{0 \leftrightarrow \infty\}) \leq \nu_+(C_\infty^+ \times \Omega) = \mu_+(C_\infty^+). \tag{10}$$

Thus, by (6) and (10) follows

$$M(\mu_+) = \phi_p^1(0 \leftrightarrow \infty) \leq \mu_+(C_\infty^+) = R(+; \mu_+). \tag{11}$$

Similarly, taking $(-)$-boundary conditions we have $|M(\mu_-)| \leq R(-;\, \mu_-)$. □

Now, we prove that below the critical temperature the percolation probability is strictly larger than the magnetization, and we give a characterization of phase transition for $d = 2$.

Theorem 3.2. *For a ferromagnetic Ising model on* $(\Sigma, \mathcal{F}, \mu_\pm)$*, at zero external field, the following relations hold:*

(i) $R(\pm;\, \mu_\pm) \geq |M(\mu_\pm)| + \dfrac{1}{2}|M(\mu_\pm)| \left(\dfrac{p}{2-p}\right)^{2d(3^d-1)} (1-p)^{2d}$, *where* $p = 1 - \exp\{-2\beta\}$*;*

(ii) if $d = 2$*, then*

$$R(\pm;\, \mu_\pm) > 0 \Leftrightarrow |M(\mu_\pm)| > 0.$$

Proof. We prove claim (i) first. Let

$$\Lambda' = \{x \in \mathbb{Z}^d \setminus \{0\} : -1 \leq x_i \leq 1 \text{ for all } i = 1, \ldots, d\}.$$

We consider the following cylinders on Λ'

$$A = \{\omega \in \Omega : \omega(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda'},\, \omega(e) = 0 \text{ for } e = \langle 0, y\rangle, y \in \Lambda'\},$$
$$B = \{\omega \in \Omega : \omega(e) = 1 \text{ for } e \in \mathbb{E}_{\Lambda'},\, \omega(e) = 1 \text{ for } e = \langle 0, y\rangle, y \in \Lambda'\}.$$

Note that the event A (resp. B) forces all edges in Λ' to be open (resp. open), and all edges incidents at the origin to be closed (resp. open).

Let x_0 be a vertex adjacent to the origin

$$\begin{aligned}\{\sigma_0 = +1\} \times (\{x_0 \leftrightarrow \infty\} \cap A) = &[(C^+_\infty \cap \{\sigma_0 = +1\}) \times (\{x_0 \leftrightarrow \infty\} \cap A)] \\ &\cup[(\overline{C^+_\infty} \cap \{\sigma_0 = +1\}) \times (\{x_0 \leftrightarrow \infty\} \cap A)],\end{aligned}$$

and it is clear that

$$\nu_+((\overline{C^+_\infty} \cap \{\sigma_0 = +1\}) \times (\{x_0 \leftrightarrow \infty\} \cap A)) = 0.$$

Thus, by using (8) and noting that events $\{\sigma_0 = +1\} \times (\{x_0 \leftrightarrow \infty\} \cap A)$ and $\Sigma \times \{0 \leftrightarrow \infty\}$ are disjoint, we obtain

$$\nu_+(C^+_\infty \times \Omega) \geq \nu_+(\Sigma \times \{0 \leftrightarrow \infty\}) + \frac{1}{2}\nu_+(\Sigma \times (\{x_0 \leftrightarrow \infty\} \cap A)),$$

hence,

$$\begin{aligned}R(+;\mu_+) = \mu_+(C^+_\infty) \geq &\ \phi^1_p(0 \leftrightarrow \infty) + \frac{1}{2}\phi^1_p(\{x_0 \leftrightarrow \infty\} \cap A) = \\ = &\ M(\mu_+) + \frac{1}{2}\phi^1_p(\{x_0 \leftrightarrow \infty\} \cap A).\end{aligned} \tag{12}$$

For the structure of the random cluster measure we obtain

$$\phi^1_p(x_0 \leftrightarrow \infty \,|\, A) = \phi^1_p(x_0 \leftrightarrow \infty \,|\, B). \tag{13}$$

Events $\{x_0 \leftrightarrow \infty\}$ and B are increasing, thus by FKG inequality [FKG71] we obtain

$$\phi^1_p(x_0 \leftrightarrow \infty \,|\, B) \geq \phi^1_p(x_0 \leftrightarrow \infty). \tag{14}$$

By (13) and (14) follows

$$\phi^1_p(\{x_0 \leftrightarrow \infty\} \cap A) \geq \phi^1_p(x_0 \leftrightarrow \infty)\,\phi^1_p(A) = M(\mu_+)\phi^1_p(A), \tag{15}$$

where the last equality follows by the translation invariance of ϕ^1_p.

We prove now a lower bound for $\phi^1_p(A)$. Let $k(d,\Lambda')$ the number of edges of the graph $(\Lambda', \mathbb{E}_{\Lambda'})$. For $\omega_{\setminus e} \in \Omega_{\setminus e} = \{0,1\}^{\mathbb{E}^d \setminus \{e\}}$, we have [For72] (see also [Gri04, New97])

$$\phi^1_p(\omega(e) = 1 \,|\, \omega_{\setminus e}) \in \left\{p, \frac{p}{2-p}\right\}. \tag{16}$$

Moreover

$$\phi^1_p(A) = \phi^1_p(\{\omega(e) = 1, e \in \mathbb{E}_{\Lambda'}\} \cap \{\omega(e) = 0, e = \langle 0, y\rangle, y \in \Lambda'\}), \tag{17}$$

thus, by (16) and (17) follows

$$\phi^1_p(A) \geq \left(\frac{p}{2-p}\right)^{k(d,\Lambda')} (1-p)^{2d}. \tag{18}$$

By (15) and (18), we obtain

$$\phi_p^1(\{x_0 \leftrightarrow \infty\} \cap A) \geq M(\mu_+) \left(\frac{p}{2-p}\right)^{k(d,\Lambda')} (1-p)^{2d}. \tag{19}$$

We also give an upper bound for $k(d,\Lambda')$. The number of vertices in Λ' is $3^d - 1$ and there are at most $2d$ edges incident to each vertex in Λ', so $k(d,\Lambda') \leq 2d(3^d-1)$. Thus, by (12) and (19) follows (i) for (+)-boundary conditions. In a similar way (i) can be proved for (−)-boundary conditions.

We can now prove claim (ii). If $|M(\mu_\pm)| > 0$ then $R(\pm;\, \mu_\pm) > 0$ by (i). Conversely, we assume $M(\mu_+) = M(\mu_-) = 0$ and prove that $R(+;\mu_+) = R(-;\mu_-) = 0$. If $M(\mu_+) = 0$, then $\mu_+ = \mu_- = \mu$ because there is not phase transition(see [Lig85]). Suppose that $R(+;\mu_+) = R(+;\mu) > 0$, hence also $R(-;\mu) > 0$. Then

$$\mu(\exists\ (\infty, \pm)\text{-cluster}) \geq R(\pm;\mu) > 0.$$

As the events $\{\exists\ (\infty,+)\text{-cluster}\}$ and $\{\exists\ (\infty,-)\text{-cluster}\}$ are invariants under translation and μ is ergodic [Geo88], we have

$$\mu(\{\exists\ (\infty,+)\text{-cluster}\} \cap \{\exists\ (\infty,-)\text{-cluster}\}) = 1. \tag{20}$$

On $\mathbb{Z}^2$, under suitable conditions (see [GKR88]), an infinite (+)-cluster cannot coexists with an infinite (−)-cluster. The conditions are: translation invariance, ergodicity, FKG inequality and invariance to reflections with respect to $\hat{x}$, $\hat{y}$ axes.

These conditions are satisfied by the Ising measure μ (see [Geo88]). This fact contradicts (20), then $R(+;\mu) = R(-;\mu) = 0$. Note that this claim is proved also in [CNPR77]. We have reported this alternative proof which immediately follows by the result in [GKR88]. □

Theorem 3.2 (i) says that if the temperature is lower than the critical temperature, or equivalently if the magnetization is positive, then the percolation probability is strictly greater than the magnetization. Moreover, for $d = 2$, Theorem 3.2 (ii) gives a characterization of phase transition through percolation. We end this section with a

Remark 3.3. *The Onsager solution for the two dimensional ferromagnetic Ising model show the exact value of magnetization as a function of $\beta \in [\beta_c, \infty)$ [Ons44]. It is*

$$M(\mu_+) = \{1 - [\sinh(2\beta)]^{-4}\}^{\frac{1}{8}}. \tag{21}$$

We can re-write (21) as a function of the parameter $x = 1 - p = \exp\{-2\beta\}$ obtaining

$$M(\mu_+) = \left\{1 - \left[\frac{2x}{1-x^2}\right]^4\right\}^{\frac{1}{8}}. \tag{22}$$

Then, using Taylor expansion we obtain $m = 1 - 2x^4 + o(x^4)$, giving the magnetization for small values of the parameter temperature (small x). We do not have an explicit

formula for the percolation probability but for small x it is easy to calculate the first terms in Taylor expansion. We find

$$R(+;\mu_+) = 1 - x^4 + o(x^4). \tag{23}$$

This general relation also holds for regular graphs

$$(1 - R(+;\mu_+)) \sim 1/2(1 - M(\mu_+)) \sim x^n$$

where n is the degree of the origin.

4 N slabs percolation and magnetization

In this section we propose a conjecture for the characterization of phase transition through percolation in the case of N slabs and some partial results.

We introduce some basic definitions for slabs. Let $\mathbb{Z}^2$ be the two-dimensional lattice, and consider the set $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$, where N is an arbitrary positive integer. The set of all the edges with endvertices in $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ is denoted by $\mathbb{E}^{2,N}$.

Definition 4.1. *An* N-vertex $\mathbf{c}_{i,j}$ *of* $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ *is a vector*

$$\mathbf{c}_{i,j} = ((i, j, 0), (i, j, 1), \dots, (i, j, N-1))$$

where $i, j \in \mathbb{Z}$. An N-edge $\mathbf{e}$ *of* $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ *is formed by a couple of N-vertices* $< \mathbf{c}_{i,j}, \mathbf{c}_{l,m} >$ *where the vertices (i, j), $(l, m) \in \mathbb{Z}^2$ are adjacent.*

We put $\Sigma^{(N)} = \{-1, +1\}^{\mathbb{Z}^2 \times \{0,1,\dots,N-1\}}$, and for $\sigma \in \Sigma^{(N)}$ we indicate with $\sigma_{i,j,k} \in \{-1, +1\}$ the spin of the vertex (i, j, k). An N*-path* is an alternating sequence of N-vertices and N-edges as in the definition of path but substituting a vertex with an N-vertex and an edge with an N-edge. An N*-subset* of $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ is a subset of $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ formed by N-vertices. An N-subset Y of $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ is N*-connected* if for all pairs of N-vertices $\mathbf{c}_{i,j}, \mathbf{c}_{r,s}$ in Y there exists an N-path formed by N-vertices in Y having $\mathbf{c}_{i,j}, \mathbf{c}_{r,s}$ as terminal N-vertices. We denote with Γ the family of all finite N-connected N-subset of $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ containing the N-vertex at the origin $\mathbf{c}_{0,0}$. An N*-box* is a finite N-subset of $\mathbb{Z}^2 \times \{0, 1, \dots, N-1\}$ as in the definition of box but substituting a vertex by an N-vertex. Analogously, we define the N-boundary of $Y \in \Gamma$, which is denoted again with ∂Y for simplicity of notation.

Definition 4.2. *For $\sigma \in \Sigma^{(N)}$, a* $(\mathbf{c}^+)$-cluster *(*$(\mathbf{c}^-)$-cluster*) of σ is a maximal connected component of N-vertices $\mathbf{c_{i,j}}$ such as*

$$\sum_{k=0}^{N-1} \sigma_{i,j,k} > 0 \quad \left(\sum_{k=0}^{N-1} \sigma_{i,j,k} < 0 \right).$$

We write $(\infty, \mathbf{c}^{\pm})$-cluster for an infinite $(\mathbf{c}^{\pm})$-cluster.

We set

$$N\text{-}C_\infty^\pm = \{\sigma \in \Sigma^{(N)} : \mathbf{c}_{0,0} \in \ (\infty, \mathbf{c}^\pm)\text{- cluster of } \sigma\},$$

where $\mathbf{c}_{0,0}$ is the N-vertex at the origin. Notice that $N\text{-}C_\infty^+$ ($N\text{-}C_\infty^-$) is the event that the N-origin belongs to an infinite cluster of N-vertices with a majority of spins $+1$ (-1) on every N-vertex. Let E^+ (E^-) be the set of configurations in $\Sigma^{(N)}$such that the N-vertex at the origin has a majority of spins $+1$ (-1) in its vertices. The events E^+ and E^- are disjoint and, for odd values of N, $E^+ \cup E^- = \Sigma^{(N)}$.

Let $\mu_\pm$ be the Ising measure on $\Sigma^{(N)}$ with $(\pm)$-boundary conditions. We set up also the *vertical interactions* $J_v \equiv 1$ between spins on two adjacent vertices belonging to different slabs. The *N-percolation probability* is $R(\mathbf{c}^\pm;\, \mu_\pm) = \mu_\pm(N - C_\infty^\pm)$.

In next proposition we show that if the N-percolation probability is positive then magnetization is positive.

Proposition 4.3. *For a ferromagnetic Ising model on $(\Sigma^{(N)}, \mathcal{F}, \mu_\pm)$ at zero external field, the following relation holds:*

$$R(\mathbf{c}^\pm; \mu_\pm) > 0 \Rightarrow |M(\mu_\pm)| > 0.$$

Proof. We project the N slabs on a single lattice, $\mathbb{Z}^2$, by assigning spins $+1$ (-1) on the vertices corresponding to N-vertices with a majority of spins $+1$ (-1) and choosing spins $+1$ or -1 with probability $\frac{1}{2}$ on the remaining vertices. This construction induces a new measure $\pi_\pm$ on $\Sigma = \{-1,+1\}^{\mathbb{Z}^2}$. We note that if there exists an infinite $(\mathbf{c}^+)$-cluster in $\mathbb{Z}^2 \times \{0, 1, \ldots, N-1\}$, then there exists an infinite $(\pm)$-cluster in the new lattice. If $M(\mu_\pm) = 0$ then $M(\pi_\pm) = 0$. Similarly to Theorem 3.2 (ii) by using the result given in [GKR88] and noting that $\pi_\pm$ satisfy all the required hypotheses, follows $R(\pm;\, \pi_\pm) = 0$. Thus also $R(\mathbf{c}^\pm;\, \mu_\pm) = 0$ by the observation above.

□

The opposite implication of Proposition 4.3 will be partially proved.

Lemma 4.4. *Let $(H, \mathcal{A}, \mathbb{P})$ an arbitrary probability space. If X and Y are random variables with X symmetric and Y not negative, then*

$$\mathbb{P}(X + Y > 0) \geq \mathbb{P}(X + Y < 0).$$

Proof. Since $Y \geq 0$, $\{X > 0\} \subseteq \{X + Y > 0\}$ and $\{X + Y < 0\} \subseteq \{X < 0\}$. Thus, because of X is symmetric

$$\mathbb{P}(X + Y > 0) \geq \mathbb{P}(X > 0) = \mathbb{P}(X < 0) \geq \mathbb{P}(X + Y < 0).$$

□

The following proposition says that if there is phase transition then the probability of a majority of $+1$ spins on the N-vertex at the origin (on any fixed N-vertex) is larger than the probability of having a majority of -1 spins on such N-vertex.

Proposition 4.5. *For a ferromagnetic Ising model on $(\Sigma^{(N)}, \mathcal{F}, \mu_\pm)$ at zero external field $|\mu_\pm(E^+) - \mu_\pm(E^-)| > 0$ if and only if $\beta > \beta_c(d)$).*

Proof. Suppose $\beta \leq \beta_c(d))$, then $\mu_+ = \mu_- = \mu$, so

$$\mu_\pm(E^+) = \mu(E^+) = \mu(E^-) = \mu_\pm(E^-).$$

Conversely, let us consider $\omega \in \Omega^{(N)} = \{0,1\}^{\mathbb{E}^{2,N}}$. Given $\omega \in \Omega^{(N)}$, the sum of spins on the vertices in $\mathbf{c}_{0,0}$ can be expressed as the sum of a symmetric random variable (vertices belonging to an finite cluster of open edges) and a positive random variable (vertices belonging to an infinite cluster of open edges), thus by Lemma 4.4

$$\nu_+(E^+ \times \Omega^{(N)} \,|\, \omega) \geq \nu_+(E^- \times \Omega^{(N)} \,|\, \omega). \tag{24}$$

We have

$$\mu_+(E^+) = \int_{\Omega^{(N)}} \nu_+(E^+ \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega), \tag{25}$$

$$\mu_+(E^-) = \int_{\Omega^{(N)}} \nu_+(E^- \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega) \tag{26}$$

Let A be the event that all vertices in $\mathbf{c}_{0,0}$ belong to an infinite open cluster. More precisely

$$A = \{\omega \in \Omega^{(N)} : (0,0,k) \leftrightarrow \infty \text{ for all } k = 0, \ldots, N-1\}.$$

Given $\omega \in A$, the conditional measure is obtained by setting $\sigma_{0,0,k} = +1$ for every $k = 0, \ldots, N-1$, thus $\nu_+(E^+ \times \Omega^{(N)} \,|\, \omega) = 1$. Hence, by (24) and (25), follows

$$\begin{aligned} \mu_+(E^+) &= \int_A \nu_+(E^+ \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega) + \int_{\Omega^{(N)} \setminus A} \nu_+(E^+ \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega) = \\ &= \int_A \phi_p^1(d\omega) + \int_{\Omega^{(N)} \setminus A} \nu_+(E^+ \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega) \geq \\ &\geq \phi_p^1(A) + \int_{\Omega^{(N)} \setminus A} \nu_+(E^- \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega). \end{aligned} \tag{27}$$

Moreover if $\omega \in A$ then $\nu_+(E^- \times \Omega^{(N)} \,|\, \omega) = 0$. So, by (26) follows

$$\mu_+(E^-) = \int_{\Omega^{(N)} \setminus A} \nu_+(E^- \times \Omega^{(N)} \,|\, \omega)\, \phi_p^1(d\omega). \tag{28}$$

By using (24), (27) and (28), we obtain

$$\mu_+(E^+) - \mu_+(E^-) \geq \phi_p^1(A). \tag{29}$$

Consider now the events

$$\begin{aligned} &F = \{\omega \in \Omega^{(N)} : (0,0,0) \leftrightarrow \infty\}, \\ &G = \{\omega \in \Omega^{(N)} : \omega(e) = 1 \text{ for } e = \langle (0,0,k-1), (0,0,k) \rangle,\ k = 1, .., N-1\}. \end{aligned}$$

We note that $A \supseteq F \cap G$. Since F and G are increasing events, by FKG inequality we obtain

$$\phi_p^1(A) \geq \phi_p^1(F \cap G) \geq \phi_p^1(F)\, \phi_p^1(G). \tag{30}$$

But, by hypothesis, $\phi_p^1(F) = \phi_p^1((0,0,0) \leftrightarrow \infty) = M(\mu_+) > 0$ and $\phi_p^1(G) > 0$ depending on a finite number of edges. By inequality (30) we get $\phi_p^1(A) > 0$, hence

$$\mu_+(E^+) - \mu_+(E^-) \geq \phi_p^1(A) > 0.$$

The same argument holds for $(-)$-boundary conditions, therefore

$$\text{phase transition} \Leftrightarrow |M(\mu_\pm)| > 0 \Rightarrow |\mu_\pm(E^+) - \mu_\pm(E^-)| > 0.$$

$\square$

We are now in the position to present our conjecture for the characterization of phase transition in the Ising model defined on the lattices $\mathbb{Z}^2 \times \{0, 1, \ldots, N-1\}$. We believe that, for these models, N-percolation probability is positive if and only if there is phase transition. Proposition 4.3 shows that an implication is true. To prove the other one we should use Proposition 4.5 and the next argument.

Let $Y \in \Gamma$ be a fixed element of Γ and we set

$$C_Y^\pm = \{\sigma \in \Sigma^{(N)} : Y \in \Gamma \text{ is a } (\mathbf{c}^\pm)\text{-cluster of } \sigma\}. \tag{31}$$

We have, as in [CNPR76]

$$\begin{aligned} \mu_+(E^+) - \mu_+(E^-) &= \mu_+(E^+) - \mu_-(E^+) = \qquad (32) \\ &= \sum_{Y \in \Gamma} (\mu_+(C_Y^+) - \mu_-(C_Y^+)) + \mu_+(N\text{-}C_\infty^+) - \mu_-(N\text{-}C_\infty^+). \end{aligned}$$

Thus, a sufficient condition for the claim to hold is that

$$\mu_+(C_Y^+) \leq \mu_-(C_Y^+), \qquad \text{for all } Y \in \Gamma. \tag{33}$$

Indeed, by assumption (33) and Proposition 4.5 for $\beta > \beta_c$ one obtains

$$R(\mathbf{c}^+; \mu_+) = \mu_+(N\text{-}C_\infty^+) \geq \mu_+(E^+) - \mu_+(E^-) > 0. \tag{34}$$

Therefore Proposition 4.3 and inequality (34) give a characterization of phase transition through percolation in the case of N slabs. In next section, we present the cases of two ($N = 2$) and three slabs ($N = 3$) with periodic boundary conditions, showing that (33) holds.

5 Two particular cases

In this section, we give a characterization of phase transition through percolation in the cases of two and three slabs. We manage to the case of two slabs the result of Theorem 3.1, the proof being similar to [CNPR76]. The extension to the case of three slabs is done in a different flavor. We start with some new definitions.

For a fixed $Y \in \Gamma$, consider an N-box Λ_o such that $\Lambda_o \supset Y \cup \partial Y$. Let $\Lambda = \Lambda_o \cup \partial\Lambda_o$ and let μ_Λ be the Ising measure on $\Sigma_\Lambda^{(N)} = \{-1, +1\}^\Lambda$ with free boundary conditions. Set

$$B^+ = \{\sigma \in \Sigma_\Lambda^{(N)} : \sigma_{i,j,0} = +1, \ldots, \sigma_{i,j,N-1} = +1 \text{ for all } \mathbf{c}_{i,j} \in \partial\Lambda_o\},$$

and similarly for B^-. If C_Y^+ is given by (31), let

$$\partial C_Y^+ = \{\sigma \in \Sigma_\Lambda^{(N)} : \sum_{l=0}^{N-1} \sigma_{i,j,l} \le 0 \text{ for each } \mathbf{c}_{i,j} \in \partial Y\} \tag{35}$$

be the set of configurations in $\Sigma_\Lambda^{(N)}$ such that each N-vertex of ∂Y does not have a majority of $+1$ spins on its vertices. In general we denote by

$$\sigma_V = \{\tilde\sigma \in \Sigma_\Lambda^{(N)} : \tilde\sigma_{i,j,k} = \sigma_{i,j,k} \text{ for all } \mathbf{c}_{i,j} \in V,\ k = 0,\dots,N-1\}$$

a cylinder where the values of $\sigma_{i,j,k} \in \{-1,+1\}$ are assigned on $V \subset \Lambda$. If V_1 and V_2 are two disjoint sets of vertices, we sometime denote by $(\sigma_{V_1}, \sigma_{V_2})$ the cylinder $\sigma_{V_1 \cup V_2}$.

We now give an inequality that will be useful in Theorem 5.1. If σ_X and $\bar\sigma_X$ are finite-dimensional cylinders with $(\bar\sigma_X)_u \ge (\sigma_X)_u$ for every vertex $u \in X$, then the following relations hold (see [Lig85]):

$$\mu_\Lambda(B^+ \,|\, \sigma_X) \le \mu_\Lambda(B^+ \,|\, \bar\sigma_X), \qquad \mu_\Lambda(B^- \,|\, \sigma_X) \ge \mu_\Lambda(B^- \,|\, \bar\sigma_X). \tag{36}$$

Theorem 5.1. *For a ferromagnetic Ising model on $(\Sigma^{(2)}, \mathcal{F}, \mu_\pm)$ at zero external field the inequality $|M(\mu_\pm)| \le R(\mathbf{c}^\pm; \mu_\pm)$ holds. Moreover $R(\mathbf{c}^\pm; \mu_\pm) > 0$ if and only if $|M(\mu_\pm)| > 0$.*

Proof. For a fixed $Y \in \Gamma$, take a N-box $\Lambda_0 \supset Y \cup \partial Y$. Let consider the cylinder $\sigma_Y \supset C_Y^+$ that we will denote with $\mathbf{1}_Y$ that assigne $+1$ spins to all the 2-vertices belonging to Y. There exists also a family $\{\sigma_{\partial Y}\}$ of cylinders $\sigma_{\partial Y} \subset \partial C_Y^+$, moreover the cylinders $\{(\mathbf{1}_Y, \sigma_{\partial Y})\}_{\sigma_{\partial Y} \subset \partial C_Y^+}$ on the vertices $(Y \cup \partial Y)$ are disjoint and $\bigcup_{\sigma_{\partial Y} \subset \partial C_Y^+} (\mathbf{1}_Y, \sigma_{\partial Y}) = C_Y^+$.

Then we can write

$$\begin{aligned} \mu_{\Lambda_o}^+(C_Y^+) &= \frac{\mu_\Lambda(C_Y^+ \cap B^+)}{\mu_\Lambda(B^+)} = \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda((\mathbf{1}_Y, \sigma_{\partial Y}) \cap B^+) = \\ &= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, (\mathbf{1}_Y, \sigma_{\partial Y})) \mu_\Lambda((\mathbf{1}_Y, \sigma_{\partial Y})) = \\ &= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y}). \end{aligned}$$

where we are using Markov property in the last equality.

Similarly

$$\mu_{\Lambda_o}^-(C_Y^+) = \frac{1}{\mu_\Lambda(B^-)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^- \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y}).$$

Since $\mu_\Lambda(B^+) = \mu_\Lambda(B^-)$, we have

$$\frac{\mu_{\Lambda_o}^+(C_Y^+)}{\mu_{\Lambda_o}^-(C_Y^+)} = \frac{\sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y})}{\sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^- \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\mathbf{1}_Y \,|\, \sigma_{\partial Y})\, \mu_\Lambda(\sigma_{\partial Y})} \le \sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})}{\mu_\Lambda(B^- \,|\, \sigma_{\partial Y})}. \tag{37}$$

Let us define the set

$$F_{\partial Y} := \{\tilde{\sigma} \in \Sigma^{(2)} : (\tilde{\sigma}_{i,j,0}, \tilde{\sigma}_{i,j,1}) \in L\} \subset \partial C_Y^+, \tag{38}$$

where $L = \{(-1,1),(1,-1)\}$ (we are not considering $(-1,-1)$). Using (36) it is clear that the supremum in (37) is achieved for $\sigma_{\partial Y} \subset F_{\partial Y}$. Let us define the operator $R : \Sigma \to \Sigma$ as:

$$(R\sigma)_{i,j,1} = \sigma_{i,j,0} \quad \text{and} \quad (R\sigma)_{i,j,0} = \sigma_{i,j,1}.$$

The following equality is clear

$$\mu_\Lambda(B^+|\sigma_{\partial Y}) = \mu_\Lambda(B^+|(R\sigma)_{\partial Y}) \tag{39}$$

because the first and second slab play the same role in the Ising measure. Moreover if $\sigma_{\partial Y} \subset F_{\partial Y}$ then $(R\sigma)_{\partial Y} = -\sigma_{\partial Y}$, and in general is $\mu_\Lambda(B^+|\sigma_{\partial Y}) = \mu_\Lambda(B^-| -\sigma_{\partial Y})$.

Thus for each $\sigma_{\partial Y} \subset F_{\partial Y}$

$$\mu_\Lambda(B^+|\sigma_{\partial Y}) = \mu_\Lambda(B^+|(R\sigma)_{\partial Y}) = \mu_\Lambda(B^+| - \sigma_{\partial Y}) = \mu_\Lambda(B^-|\sigma_{\partial Y}). \tag{40}$$

Hence, by (40) and previous argument

$$\sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \mid \sigma_{\partial Y})}{\mu_\Lambda(B^- \mid \sigma_{\partial Y})} = \sup_{\sigma_{\partial Y} \subset F_{\partial Y}} \frac{\mu_\Lambda(B^+ \mid \sigma_{\partial Y})}{\mu_\Lambda(B^- \mid \sigma_{\partial Y})} = 1. \tag{41}$$

Since this relation holds for all $\Lambda_o \supset Y \cup \partial Y$ then also in the limit $\Lambda_o \to \mathbb{Z}^2 \times \{0, 1\}$, by (37) and (41) we obtain (33).

To prove the inequality between 2-percolation probability and magnetization it is enough to observe that, by symmetry, we have $\mathbb{E}_{\mu_\pm}(\sigma_{0,0,0}) = \mathbb{E}_{\mu_\pm}(\sigma_{0,0,1})$, hence

$$\begin{aligned} M(\mu_\pm) &= \frac{1}{2}\, \mathbb{E}_{\mu_\pm}(\sigma_{0,0,0} + \sigma_{0,0,1}) = \\ &= \sum_{\overline{\sigma}_0, \overline{\sigma}_1 \in \{-1,\, +1\}} \frac{1}{2}\, (\overline{\sigma}_0 + \overline{\sigma}_1)\, \mu_\pm(\sigma_{0,0,0} = \overline{\sigma}_0,\, \sigma_{0,0,1} = \overline{\sigma}_1) = \\ &= \mu_\pm(E^+) - \mu_\pm(E^-). \end{aligned}$$

Now, the first claim of the theorem immediately follows by (32) and (33). The second claim follows by the first inequality and Proposition 4.3.

□

We present another particular case, in which we are able to prove (33), and thus to obtain characterization of phase transition via percolation. We consider the graph $\tilde{\mathcal{G}}_3$ having vertex set $\mathbb{Z}^2 \times \{0,1,2\}$ and edge set $\mathbb{E}^{2,3} \cup \mathbb{E}^p$, where

$$\mathbb{E}^p = \{\langle (i,j,0),(i,j,2)\rangle : i,j \in \mathbb{Z}\}.$$

Consider on $\tilde{\mathcal{G}}_3$ the ferromagnetic Ising measures $\mu_\pm$, and define

$$D_Y^+ = \{\sigma \in \Sigma^{(N)} : Y \in \Gamma \text{ belongs to a } (\mathbf{c}^+) - \text{cluster of } \sigma\}, \tag{42}$$

so that $C_Y^+ = D_Y^+ \cap \partial C_Y^+$. The event D_Y^+ depends only on the values of $\{\sigma_{i,j,k} : \mathbf{c}_{i,j} \in Y,\, k = 0, \ldots, N-1\}$.

Theorem 5.2. *For a ferromagnetic Ising model on $(\Sigma^{(3)}, \mathcal{F}, \mu_\pm)$, at zero external field $R(\mathbf{c}^\pm; \mu_\pm) > 0$ if and only if $|M(\mu_\pm)| > 0$.*

Proof. If $R(\mathbf{c}^\pm; \mu_\pm) > 0$, then $|M(\mu_\pm)| > 0$ by Proposition 4.3 with $N = 3$. Conversely, if $|M(\mu_\pm)| > 0$, we can use Proposition 4.5 and prove that (33) holds. Indeed, for a fixed $Y \in \Gamma$, consider the set of all cylinders $\sigma_Y \subset D_Y^+$ and $\sigma_{\partial Y} \subset \partial C_Y^+$. Then, using Markov property, we have

$$\begin{aligned}
\mu_{\Lambda_o}^+(C_Y^+) &= \frac{\mu_\Lambda(C_Y^+ \cap B^+)}{\mu_\Lambda(B^+)} = \\
&= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+, \sigma_Y \subset D_Y^+} \mu_\Lambda((\sigma_Y, \sigma_{\partial Y}) \cap B^+) = \\
&= \frac{1}{\mu_\Lambda(B^+)} \sum_{\sigma_{\partial Y} \subset \partial C_Y^+} \mu_\Lambda(B^+ \,|\, \sigma_{\partial Y}) \mu_\Lambda(\sigma_{\partial Y}) \sum_{\sigma_Y \subset D_Y^+} \mu_\Lambda(\sigma_Y \,|\, \sigma_{\partial Y}),
\end{aligned}$$

and similarly for $\mu_{\Lambda_o}^-(C_Y^+)$, hence:

$$\frac{\mu_{\Lambda_o}^+(C_Y^+)}{\mu_{\Lambda_o}^-(C_Y^+)} \leq \sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})}{\mu_\Lambda(B^- \,|\, \sigma_{\partial Y})}.$$

We need to prove that

$$\sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})}{\mu_\Lambda(B^- \,|\, \sigma_{\partial Y})} \leq 1. \tag{43}$$

Let us define

$$G_{\partial Y} := \{\tilde{\sigma} \in \Sigma : (\tilde{\sigma}_{i,j,0}, \tilde{\sigma}_{i,j,1}, \tilde{\sigma}_{i,j,2}) \in L\} \subset \partial C_Y^+, \tag{44}$$

where $L = \{(-1,-1,1), (-1,1,-1), (1,-1,-1), \}$. Relations (36) shows that the supremum in (43) is achieved on cylinders that are subset of $G_{\partial Y}$.

For total spin flip invariance

$$\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y}) = \mu_\Lambda(B^- \,|\, -\sigma_{\partial Y}). \tag{45}$$

We now define the rotation operator $R : \Sigma \to \Sigma$ as:

$$(R\sigma)_{i,j,k} = \sigma_{i,j,k+1} \qquad \forall (i,j,k) \in \mathbb{Z}^2 \times \{0,1,2\}, k = 0,1,2 \tag{46}$$

where $\sigma_{i,j,0} = \sigma_{i,j,3}$. Since μ_Λ is invariant under rotation of the three slabs, we have

$$\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y}) = \mu_\Lambda(B^+ \,|\, (R\sigma)_{\partial Y}) = \mu_\Lambda(B^- \,|\, (-R\sigma)_{\partial Y}). \tag{47}$$

Now observe that if $\mathbf{c}_{i,j} \in \partial Y$, then $\sigma_{i,j,k} \leq (-R\sigma)_{i,j,k}$ holds for all $\sigma \in G_{\partial Y}$, hence

$$\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y}) = \mu_\Lambda(B^- \,|\, (-R\sigma)_{\partial Y}) \leq \mu_\Lambda(B^- \,|\, \sigma_{\partial Y}), \tag{48}$$

and

$$\limsup_{\Lambda \uparrow \mathbb{Z}^2 \times \{0,1,2\}} \sup_{\sigma_{\partial Y} \subset \partial C_Y^+} \frac{\mu_\Lambda(B^+ \,|\, \sigma_{\partial Y})}{\mu_\Lambda(B^- \,|\, \sigma_{\partial Y})} \leq 1, \tag{49}$$

implying (33). This concludes the proof. □

Theorem 5.2 says that there exists a phase transition in the Ising model on $\Sigma^{(3)}$ if and only if there is a positive probability that the 3-vertex at the origin belongs to an infinite cluster of 3-vertices with a majority of $+1$ spins on its vertices. Contrary to the case of two slabs, we do not obtain an inequality between the 3-percolation probability and magnetization since in the case of three slabs we cannot write

$$M(\mu_\pm) = \mu_\pm(E^+) - \mu_\pm(E^-).$$

A natural problem to address is that of determining whether there exists a maximal number of slabs for which the only extremal Gibbs measures are μ_+ and μ_-.

Problem 5.3. *Define*

$$N_c = \sup\{N \in \mathbb{N} : \textit{ the only extremal measures on } N\textit{-slabs are } \mu_+ \textit{ and } \mu_-\}.$$

Two natural questions are: is N_c finite or infinite? Is N_c equal to one? We conjecture that $N_c = \infty$. Hence the behavior in three dimensions should remain meaningfully different with respect to slab graphs.

We end the paper with some remarks.

Throughout the paper we have only considered constant interactions equal to 1. However, one can see that all proofs work similarly if different values of the interactions on the slabs are chosen: J_o interaction between spins on the same level and J_v vertical interaction between spins on different slabs. The symmetries between the slabs in Theorem 5.1 and Theorem 5.2 still hold and so the proofs work without modifications.

A second remark concerns a possible interpretation of Proposition 4.5. Indeed, Proposition 4.5 can be used in order to obtain a filtering result; let us suppose that a configuration $\sigma \in \Sigma^{(N)}$ generated by the measure μ_+ is represented only by giving the following information: on the N-vertex v there is a $+1$ majority, a -1 majority or the same proportion of $+1$ and -1. By using Proposition 4.5 we can say that this information is sufficient to establish whether μ_+ is in a region of phase transition ($T < T_c$) or not ($T \geq T_c$). It's enough to observe that, on a sequence of boxes invading all the space, the frequency of N-vertices with $+1$ majority is definitively larger than the frequency of N-vertices with -1 majority if and only if there is phase transition (we are also using the ergodicity of the measure μ_+).

We present also an extension of the phase transition characterization via percolation to some exotic graphs. We only give an example of these graphs in which the result can be applied. Let's consider $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$. For each vertex $v = (i, j) \in \mathbb{Z}^2$ we take a number n_0 of vertices denoted by $(v; l) = (i, j; l)$ for $l = 1, \ldots, n_0$. The set $H_v = \{(v; l), l = 1, \ldots, n_0\}$ is called *hyper-vertex*. We put an edge between all pairs of vertices $(v; l), (v; m), l, m = 1, \ldots, n_0$; moreover for $e =< u, v >\in \mathbb{E}^2$ we set an edge between each pair of vertices $(u; l), (v; m), l, m = 1, \ldots, n_0$. Now let's define the Ising model with plus boundary conditions on this graph and declare that the *hyper-spin* S_v on the hyper-vertex H_v is equal to $\text{sign}(\sum_{l=1}^{n_0} \sigma_{v;l})$, where $\text{sign}(0) = 0$. For the random field $\{S_v\}_{v\in\mathbb{Z}^2}$ there is percolation (*i.e.* there exists an infinite cluster of hyper-vertices with plus hyper-spins) if and only if μ_+ is in the phase transition region ($T < T_c$).

References

[Aiz80] M. Aizenman. Translation invariance and instability of phase coexistence in the two-dimensional Ising system. *Comm. Math. Phys.*, 73(1):83–94, 1980.

[BB97] B. Bollobás and G. Brightwell. Random walks and electrical resistances in products of graphs. *Discrete Appl. Math.*, 73(1):69–79, 1997.

[Bod03] T. Bodineau. Slab percolation for the ising model. *arXiv:math.PR/0309300*, pages 1–33, 2003.

[CNPR76] A. Coniglio, C.R. Nappi, F. Peruggi, and L. Russo. Percolation and phase transitions in the Ising model. *Comm. Math. Phys.*, 51(3):315–323, 1976.

[CNPR77] A. Coniglio, C.R. Nappi, F. Peruggi, and L. Russo. Percolation points and critical point in the Ising model. *J. Phys. A*, 10(2):205–218, 1977.

[CR85] M. Campanino and L. Russo. An upper bound on the critical percolation probability for the three-dimensional cubic lattice. *Ann. Probab.*, 13(2):478–491, 1985.

[FKG71] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22:89–103, 1971.

[For72] C. M. Fortuin. On the random-cluster model. III. The simple random-cluster model. *Physica*, 59:545–570, 1972.

[Geo88] H.-O. Georgii. *Gibbs measures and phase transitions*, volume 9 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1988.

[GH00] H.-O. Georgii and Y. Higuchi. Percolation and number of phases in the two-dimensional Ising model. *J. Math. Phys.*, 41(3):1153–1169, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[GKR88] A. Gandolfi, M. Keane, and L. Russo. On the uniqueness of the infinite occupied cluster in dependent two-dimensional site percolation. *Ann. Probab.*, 16(3):1147–1157, 1988.

[Gri04] G. Grimmett. The random-cluster model. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 73–123. Springer, Berlin, 2004.

[Häg03] O. Häggström. Probability on bunkbed graphs. *Formal Power Series and Algebraic Combinatorics*, pages 19–27, 2003.

[Hig81] Y. Higuchi. On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 517–534. North-Holland, Amsterdam, 1981.

[Hig85] Y. Higuchi. A weak version of RSW theorem for the two-dimensional Ising model. In *Particle systems, random media and large deviations (Brunswick, Maine, 1984)*, volume 41 of *Contemp. Math.*, pages 207–214. Amer. Math. Soc., Providence, RI, 1985.

[Hig87] Y. Higuchi. Percolation of the two-dimensional Ising model. In *Stochastic processes—mathematics and physics, II (Bielefeld, 1985)*, volume 1250 of *Lecture Notes in Math.*, pages 120–127. Springer, Berlin, 1987.

[Hig93a] Y. Higuchi. Coexistence of infinite (∗)-clusters. II. Ising percolation in two dimensions. *Probab. Theory Related Fields*, 97(1-2):1–33, 1993.

[Hig93b] Y. Higuchi. A sharp transition for the two-dimensional Ising percolation. *Probab. Theory Related Fields*, 97(4):489–514, 1993.

[Lig85] T.M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[New97] C.M. Newman. *Topics in disordered systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1997.

[Ons44] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, 65:117–149, 1944.

[Rus79] L. Russo. The infinite cluster method in the two-dimensional Ising model. *Comm. Math. Phys.*, 67(3):251–266, 1979.